Solution to Assignment 4

15.5

(24). The region is over a rectangle which can be decomposed into two trianlges D_1 and D_2. D_1 has vertices at $(0,0),(1,0),(1,2)$ and D_2 has vertices at $(0,0),(1,2),(0,2)$. Over D_1, the region is described by $0 \le z \le 1-x$. Over D_2, it is given by $0 \le z \le (2-y)/2$. Hence the volume of the region is

$$\iint_{D_1} \int_0^{1-x} 1\,dz\,dA(x,y) + \iint_{D_2} \int_0^{(2-y)/2} 1\,dz\,dA(x,y) \ = \cdots \ .$$

(27). The equation of the plane passing through $(1,0,0),(0,2,0),(0,0,3)$ is given by $6x+3y+2z = 6$ (after using the cross product method). Regarding it as a region over the triangle T in the xy-plane with vertices at $(0,0),(1,0),(0,2)$, the volume of the tetrahedron is

$$\iint_T \int_0^{(6-6x-3y)/2} dz\,dA(x,y) = \cdots \ .$$

(29) The region is described by $0 \le z \le \sqrt{1-x^2}$ where (x,y) satisfies $x^2+y^2 \le 1, x,y \ge 0$. Therefore, the volume of this region is

$$8 \times \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} 1\,dz\,dydx = \cdots = 16/3 \ .$$

Supplementary Problems

1. Find the equations of the planes passing through the origin and (a) $(1,2,3),(0,-2,0)$ and (b) $(0,2,-1),(3,0,5)$.

   Solution. (a) $(1,2,3)\times(0,-2,0) = (6,0,-2)$. The equation is $6x-2z=0$ or $3x-z=0$.

   (b) $(0,2,-1)\times(3,0,5) = (10,-3,-6)$. The equation is $10x-3y-6z=0$.

2. Find the equation of the plane passing the points $(1,0,-1),(4,0,0),(6,2,1)$.

   Soluton. Take $\mathbf{u}_0 = (4,0,0)$. Then $\mathbf{v}_1 = (1,0,-1)-(4,0,0) = (-3,0,-1)$, and $\mathbf{v}_2 = (6,2,1)-(4,0,0) = (2,2,1)$. $\mathbf{v}_1\times\mathbf{v}_2 = (2,1,-6)$. The equation is $2x+y-6z=d$. Since $(4,0,0)$ belongs to the plane, $d = 2\times 4+0-6\times 0 = 8$. Finally, the equation of this plane is $2x+y-6z=8$.

3. Let C be the cone whose top is $(0,0,h)$ and base is a triangle T in the xy-plane. Show that its volume is given by $\frac{1}{3}|T|h$ where $|T|$ is the area of T.

   Solution. Use the volume formula involving the cross sections. At z, the cross section of the paraboloid is given by $(x,y), x^2+y^2 \le z$. This is a disk of radius $\sqrt{z}$ whose area is πz. So the volume of the paraboloid is

   $$\int_0^h \pi z\,dz = \frac{\pi h^2}{2} \ .$$